An Elementary Proof of the Irrationality of e

We present a proof of the irrationality of e that requires (almost) no calculus. Motivated by $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = 1 + \frac{1}{1}(1 + \frac{1}{2}(1 + \frac{1}{3}(1 + \cdots)))$, let's define a sequence $\{x_n\}_{n\in\mathbb{N}}$ by

$$x_n = \frac{1}{n} + \frac{1}{n(n+1)} + \frac{1}{n(n+1)(n+2)} + \cdots .$$

The following properties are obvious:

1. $e = 1 + x_1 = 1 + \frac{1}{1}(1 + x_2) = 1 + \frac{1}{1}(1 + \frac{1}{2}(1 + x_3)) = \cdots$.
2. $x_n = \frac{1}{n}(1 + x_{n+1})$.
3. $x_1 > x_2 > x_3 > \cdots > 0$.

Assuming that e is rational, so are all x_n because of Property 1. Let $x_n = \frac{p_n}{q_n}$ where p_n and q_n are relatively prime positive integers. From Property 2, $\frac{p_n}{q_n} = \frac{1}{n}\left(1 + \frac{p_{n+1}}{q_{n+1}}\right)$, so $\frac{p_{n+1}}{q_{n+1}} = \frac{np_n - q_n}{q_n}$. Thus $q_n \geq q_{n+1}$, i.e., $q_1 \geq q_2 \geq q_3 \geq \cdots$. Combining this with Property 3, $\frac{p_1}{q_1} > \frac{p_2}{q_2} > \frac{p_3}{q_3} > \cdots > 0$; we conclude that $p_1 > p_2 > p_3 > \cdots > 0$, i.e., $\{p_n\}$ is a strictly decreasing infinite sequence of positive integers. This is impossible.

Remark. Unlike other existing proofs based on the series definition of e, this proof does not require any series analysis as long as we accept that e is well-defined. It actually shares the flavor of the continued fraction approach (e.g., [**1**], [**2**], and [**3**, pp. 185–190]).

REFERENCES

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